A Scaling Law for Planar and Helical Superconducting Undulators

SRI 2010 Workshop 3: Superconducting undulators and other new ID sources September 20 – 21, 2010 at the APS Conference Center

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**Introduction**

Iron dominated magnet [1]

\[
N_l \sim \frac{1}{\mu} \oint B \cdot d\vec{l} \rightarrow B(g)_{D,pole} \approx \frac{\mu_0 N_l}{g/2}
\]

\[
B(r_0)_{Quad} \approx \frac{2\mu_0 N_l}{r_0}
\]

\[
B(r_0)_{Sext} \approx \frac{3\mu_0 N_l}{r_0}
\]

\[
B_0 = \text{const} \cdot j
\]

Electromagnetic undulator [2]

\[
B_n(\mu \approx \infty) = \frac{32\mu_0 N_l}{\pi n} \frac{\sin(n\pi/4)}{\sinh(nkg/2)} (k = \frac{2\pi}{\lambda})
\]

\[
B_y = \frac{32\mu_0 N_l}{\sqrt{2}\pi \lambda} \left[ \frac{\cosh(ky)}{\sinh(kg/2)} - \frac{\cosh(3ky)}{3\sinh(3kg/2)} \right]
\]

Permanent magnet undulator [3]

\[
B_0 = 2B_r \frac{\sin(\pi/m)}{(\pi/m)} (1 - e^{-kh}) e^{-kg/2}
\]

\[
B_0 = a \cdot \exp \left[ -b \frac{g}{\lambda} + c \left( \frac{g}{\lambda} \right)^2 \right], \quad (0.07 < \frac{g}{\lambda} < 0.7)
\]

SmCo5: Br = 0.9 T, a = 3.33, b = 5.47, c = 1.8
NdFeB: Br = 1.1 T, a = 3.44, b = 5.08, c = 1.54

Planar superconducting undulator

![Graph showing peak field vs. coil current density for different wavelengths and gap-to-λ ratios.](image)

**References**

Outline

- Introduction
- Analytical expression of planar (electromagnetic) undulator
- $j\lambda$ scaling and Ampere’s law
- Data on pole-gap dependence and coil dimensions to support the $j\lambda$ scaling
- Characteristics, $j\lambda$ scaling, and analytical expression of bifilar helical undulator
- An example of the $j\lambda$ scaling for a superconducting undulator (SCU): Why it is much more difficult with “short-period” SCUs
- Summary
Planar undulator

Analytical expressions [1]

\[ j_x(z) = \sum_{n=1,3,\ldots}^{\infty} 4j \frac{n \pi}{n \pi} \sin(nk \frac{a}{2}) \cos(nkz) \quad (k = \frac{2\pi}{\lambda}) \]

per unit height of the coils

\[ B_y(y, z) = \sum_{n=1,3,\ldots}^{\infty} B_n \cosh(nky) \sin(nkz) \]

\[ B_z(y, z) = \sum_{n=1,3,\ldots}^{\infty} B_n \sinh(nky) \cos(nkz) \]

\[ B_n = 2\mu_0 j \frac{\lambda}{(n \pi)^2} \sin(nk \frac{a}{2}) e^{-nkg/2} \left(1 - e^{-nk_b}\right) \]

\[ B_y(0, z) = B_0 \sin(kz) \]

\[ B_z(0, z) = B_0 \cos(kz) \]

\[ B_0 = \frac{2\mu_0 j \lambda}{\pi^2} \sin(k \frac{a}{2}) e^{-kg/2} \left(1 - e^{-kb}\right) \]

- When SCU dimensions are scaled, with \( l \) as the reference, \( B_0 \) remains unchanged for a constant \( jl \).
- The on-axis peak field depends approximately on \( \exp(-kg/2) \).
- The derived equation \( B_n \) predicted the third harmonic of the on-axis field correctly.
- Surprisingly, the \( jl \) scale holds for undulators with nonlinear magnetic materials.

Does Ampere’s law explain the $jl$ scaling?

For a constant $jl$, if the flux density distributions remain unchanged in the above two geometries (which include the nonlinear magnetic poles and flux-return yokes), then the $jl$ scaling may be understandable with Ampere’s law.

Numerical analysis showed that the flux density, as well as the permeability, distributions of the two were identical [1].

[1] Opera, Vector Fields Software, Cobham Technical Services, Aurora, IL 60505, USA
Peak field $B_0$ dependence on $g/\lambda$ for nonlinear cases

$$B_0(j, \frac{g}{\lambda}) = B_0(j, 0.5)\exp[-\pi(\frac{g}{\lambda} - 0.5)] \times \{1 + \delta j_n^{-1}\exp[-2.4\pi(\frac{g}{\lambda} - 0.5)]\} \quad \text{Eq. (1)} \quad (\delta = 0.0016)$$

- For $g/\lambda > 0.4$, $B_0$ is proportional to $\exp(-k g/2)$ within 1%.
- The above equation, which has a correction term, holds for $g/\lambda > 0.06$ within 1%.
When the pole gap is reduced, the coil maximum field increases, which, in turn, reduces the SC critical current. Therefore, the correction term for $g/\lambda > 0.4$ in Eq. (1) may be neglected.

Bigger coil dimensions make higher coil maximum fields, which reduce the SC critical current densities. Therefore, the spread of the peak fields reduced, and the dependence of the achievable peak fields on coil dimensions is relatively small.
Helical undulator

**Helical Solenoid** [1]

\[ B_r = \frac{\mu_0 I}{\lambda} \{ kr_0 K_0(kr_0) + K_1(kr_0) \} \]

**Helical undulator** [2]

\[ B_0 = \frac{2\mu_0 I}{\lambda} \{ kr_0 K_0(kr_0) + K_1(kr_0) \} \]

For \( I_1 \) and \( I_2 \) in opposite directions in each helix:

\[ B_0 = \frac{\mu_0 (I_1 + I_2)}{\lambda} \{ kr_0 K_0(kr_0) + K_1(kr_0) \} \]

\[ B_z = \frac{\mu_0 (I_1 - I_2)}{\lambda} \]

**Helical Undulators with coil dimensions \((a, b)\)** [3]

\[
B < = \sum_{n=1,3,5,}^\infty B_0^n \cdot \left[ \hat{r} \{ I_{n-1}(nkr) + I_{n+1}(nkr) \} \sin n(kz - \phi) + \hat{\phi}(-2/nkr)I_n(nkr)\cos n(kz - \phi) + \hat{z}2I_n(nkr)\cos n(kz - \phi) \right] \\
B_0^n = \frac{2\mu_0 j}{\pi} \sin \left( \frac{nka}{2} \right) \{ kr_0 K_{n-1}(nkr_0) + K_n(nkr_0) \} \\
B(kz - \phi) = B_0 \left\{ \hat{r} \sin(kz - \phi) - \hat{\phi} \cos(kz - \phi) \right\} \\
I_m(x) = \sum_{\ell=0}^{\infty} \frac{(x/2)^{m+2\ell}}{\ell!(m+\ell)!} \quad (m \geq 1)
\]

- On-axis field has no higher harmonic fields, while the off-axis field components do.
- For a constant \( j/\lambda \), \( B_0 \) remains unchanged when length dimensions are scaled.
- How about with steel poles?

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Helical undulator:
Model calculation to verify the analytical results

- period = 12 mm, coil $r_0 = 3.15$ mm
- coil dimension $b = 3.84$ mm
- $j(\text{coil}) = 1\text{kA/mm}^2$
- $(B_{OP} - B_{Eq})/B_{Eq} < 0.1\%$

\[
B_0 = 0.8 j(\lambda) \sin\left(\frac{ka}{2}\right) \int_{r_0}^{r_0+b} [krK_0(kr) + K_1(kr)] dr \quad \frac{B_0}{T} = \frac{j[kA/mm^2]}{\lambda[mm]}
\]

- Analytical equation agrees with model coil calculations within 0.1%.
- Higher harmonic coefficients of the on-axis field are less than $10^{-6}$.

\[
[K] \approx \exp(-0.95kr)
\]
Helical undulator: Scaling law for steel poles

\[ \lambda = 12, j_0 = 1.0 \text{kA/mm}^2, B_0 = 0.877286 \text{T}, \text{ steel poles} \]
\[ \lambda = 24, j_0 = 0.5 \text{kA/mm}^2, B_0 = 0.876782 \text{T}, \text{ steel poles, } \frac{\Phi B}{B_0} = 5 \times 10^{-4} \]
\[ \lambda = 12, j_0 = 1.0 \text{kA/mm}^2, B_0 = 0.577 \text{T}, \text{ air poles} \]

- Off-axis field components follow the \( j\lambda \) scaling within 0.1%.
- On-axis field and coil maximum field for the two periods follow the \( j\lambda \) scaling.
- When the off-axis field components are normalized to the on-axis fields for the steel poles and air poles, the field components follow the \( j\lambda \) scaling for \( r < 3 \text{ mm} \).
How to use the $j \bullet$ scaling law and why it is so difficult with “short-period” planar and helical SCUs

- $\bullet = 20 \text{ mm,}$ 0.4 0.8 1.2 1.6 2.0 2.4 2.8
- $\bullet = 8 \text{ mm,}$ 1.0 2.0 3.0 4.0 5.0 6.0 7.0

![Diagram of NbTi SC coil $j_c$ at 4.2 K](image)

$\lambda = 16 \text{ mm, } g = 8 \text{ mm}

An example:
Scale from $\bullet = 16$ to $\bullet = 8 \text{ mm}$

- $B_m = 3.6 \text{ T, } j_c = 1.55 \text{ (@ } \lambda = 16\text{)}$
- $B_0 = 1.25, K = 1.868 \text{ (@ } \lambda = 16\text{)}$
- $j \lambda \rightarrow (2j)(\lambda/2):$ same $B_m, B_0$ data
- $K \rightarrow 0.5K \text{ (1.868} \rightarrow 0.934\text{)}$
- $g \rightarrow 0.5g \text{ (8 mm} \rightarrow 4 \text{ mm)}$
- $j_c$(SC)-curve $\rightarrow$ not $2j_c$(SC)-curve
- $j_c = 1.55 \rightarrow 1.8, B_0 = 1.25 \rightarrow 0.9 \text{ T}$
- $K = 0.934 \rightarrow 0.672$
- $B_0(g_1) = (B_0)\exp\{-\pi(g_1-4)/8\}$
- $B_0(6 \text{ mm}) = 0.41 \text{ T}$
- New $K = 0.305$
- All at NbTi $j_c$ at 4.2 K.

Don’t be discouraged. If you increase the period from 16 to only 20 mm, there will be big rewards!

[Graph showing coil current density ($j$) vs. peak field ($B_0$) with scaling parameters and example data points.]
The derived analytical expressions of $B_0$ for planar and helical undulators show that, when length dimensions are scaled according to a period ratio, the field remain unchanged for a constant $j_0 \lambda_0 = j_n \lambda_n$.

The $j \lambda$ scaling law extends to the distributions of the flux density and permeability for the whole region of SCUs with nonlinear poles and yokes.

With one data set for $B_0$ and $B_m$, $B_0$ for different periods may be calculated.

- $B_0$ dependence on coil dimension is insignificant.

For $g/\lambda > 0.15$, the peak field varies as

$$B_0 \left( \frac{g}{\lambda} \right) = B_0 \left( \frac{g_1}{\lambda} \right) \exp \left[ -\pi \left( \frac{g}{\lambda} - \frac{g_1}{\lambda} \right) \right]$$

As an example for using the $j \lambda$ scaling law, it was shown why achieving a “required” $B_0$ for a “shorter period” is an issue.