Investigation of Coherent X-ray Propagation and Diffraction

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Outline

1. Introduction
2. Mutual Coherence
3. ABCD Optics
4. Propagation of Coherence
5. Measurement of Coherence
6. Conclusion

Spectroscopy

$$\begin{aligned} I(\omega) &= |E(\omega)|^2 \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^*(t)E(t')e^{i\omega(t-t')}dt\,dt' \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^*(t)E(t+\tau)e^{i\omega\tau}dt\,d\tau \end{aligned}$$

Wiener-Khinchin theorem

$$I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle E^*(t)E(t+\tau)\rangle e^{i\omega\tau}d\tau$$

Correlation functions

One conventionally normalizes

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle}$$

More generally:

$$g^{(1)}(\vec{\delta r},\tau) = \frac{\langle E^*(\vec{r}+\vec{\delta r},t+\tau)E(\vec{r},t)\rangle}{\langle E^*(\vec{r},t)E(\vec{r},t)\rangle}$$

And its spatial Fourier transform is what matters for diffraction.

More correlations

Can also define a higher order correlation function:

$$g^{(2)}(\tau) = \frac{\langle I(Q,t)I(Q,t+\tau)\rangle}{\langle I(Q,t)\rangle^2} = \frac{\langle E^*(t)E(t)E^*(t+\tau)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle^2}$$

And *often* this is (Gaussian statistics):

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$$

Mutual Coherence Function

Define

$$\Gamma(\vec{r}_1,\vec{r}_2,t_1,t_2) = \langle E^*(\vec{r}_1,t_1)E(\vec{r}_2,t_2)\rangle.$$

where $\vec{E}(\vec{r},t)$ is the electric field. Also define

$$W(\vec{r}_1,\vec{r}_2,\nu) = \int_{-\infty}^{\infty}\Gamma(\vec{r}_1,\vec{r}_2,\tau)e^{2\pi i\nu\tau}d\tau$$

and it's normalized form:

$$\mu(\vec{r}_1,\vec{r}_2,\nu) = \frac{W(\vec{r}_1,\vec{r}_2,\nu)}{W(\vec{r}_1,\vec{r}_1,\nu)^{1/2}W(\vec{r}_2,\vec{r}_2,\nu)^{1/2}}.$$

Finally:

$$\langle E^*(\vec{r}_1,\nu_1)E(\vec{r}_2,\nu_2)\rangle = W(\vec{r}_1,\vec{r}_2,\nu_1)\delta(\nu_1-\nu_2).$$

Reference: Coherent X-ray Diffraction, Mark Sutton

Third-Generation Hard X-ray Synchrotron Radiation Sources,

Ed. D. M. Mills, Wiley (2002).

Why Coherence?

Coherence allows one to measure the dynamics of a material (X-ray Photon Correlation Spectroscopy, XPCS).

$$\langle I(\vec{Q},t)I(\vec{Q}+\delta\vec{\kappa},t+\tau)\rangle = \langle I(Q)\rangle^2 + \beta(\vec{\kappa})\,\frac{r_0^4}{R^4}V^2 I_0^2 \left|S(\vec{Q},t)\right|^2$$

where the coherence part is:

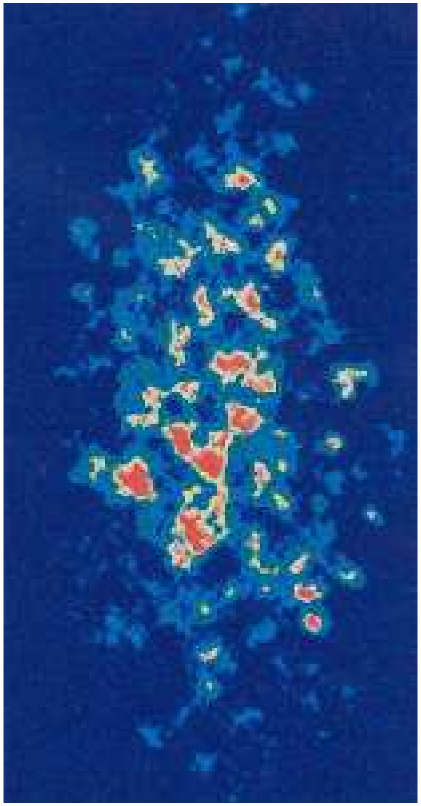
$$\beta(\vec{\kappa}) = \frac{1}{V^2 I_0^2}\int_V\int_V e^{i\vec{\kappa}\cdot(\vec{r}_2-\vec{r}_1)} \left|\Gamma(\vec{0},\vec{r}_2^{\perp}-\vec{r}_1^{\perp},\frac{\vec{Q}\cdot(\vec{r}_2-\vec{r}_1)}{\omega_0})\right|^2 d\vec{r}_1 d\vec{r}_2$$

and $\beta(\vec{0}) \approx \frac{V_{coherence}}{V_{scattering}}$ with widths $\lambda/V^{\frac{1}{3}}$

Reference: M. Sutton, Coherent X-ray Diffraction, in **Third-Generation Hard X-ray Synchrotron Radiation Sources: Source Properties, Optics, and Experimental Techniques**, edited by. Dennis M. Mills, John Wiley and Sons, Inc, New York, (2002).

Coherent diffraction

(001) Cu_3Au peak



Sutton et al., The Observation of Speckle by Diffraction with Coherent X-rays, Nature, **352**, 608-610 (1991).

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Gaussian Schell Model

Assume:

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2) = I(\mathbf{r}_1,t_1)^{\frac{1}{2}}(I(\mathbf{r}_2,t_2)^{\frac{1}{2}}\mu(\mathbf{r}_1-\mathbf{r}_2,t_2-t_1)$$

Since $\mu(0) = 1$, we see that $\Gamma(\mathbf{r},\mathbf{r},t,t) = I(\mathbf{r},t)$

Assume profiles are Gaussian (Simplifying to one dimension)

$$I(x) = \frac{I_0}{\sqrt{2\pi}\sigma_s}e^{-\frac{x^2}{2\sigma_s^2}}$$

and

$$\mu(x_2-x_1) = e^{-\frac{(x_2-x_1)^2}{2\rho_c^2}}.$$

Gaussian Schell Model

Writing the mutual coherence as a function of $r_1 + r_2$ and $\mathbf{r}_2 - \mathbf{r}_1$ gives

$$\Gamma(x_1, x_2) = \frac{I_0}{\sqrt{2\pi}\sigma_s} e^{-\frac{x_1^2}{4\sigma_s^2}} e^{-\frac{x_2^2}{4\sigma_s^2}} e^{-\frac{(x_2-x_1)^2}{2\rho_c^2}}$$

$$= \frac{I_0}{\sqrt{2\pi}\sigma_s} e^{-\frac{(x_1+x_2)^2}{8\sigma_s^2}} e^{-(x_2-x_1)^2\left(\frac{1}{2\rho_c^2}+\frac{1}{8\sigma_s^2}\right)}$$

$$= \frac{I_0}{\sqrt{2\pi}\sigma_s} e^{-\frac{(x_1+x_2)^2}{8\sigma_s^2}} e^{-\frac{(x_2-x_1)^2}{2\delta_a^2}}$$

defining

$$\frac{1}{\delta_a^2} = \frac{1}{\rho_c^2} + \frac{1}{4\sigma_s^2}$$

Brightness

Conventionally brightness has the form:

$$B(\vec{r},\hat{s},\nu) = \frac{1}{4\pi^2}\frac{I_0 H(\nu)}{\sigma_{a_h}\sigma_{a_v}\sigma_h\sigma_v} e^{-(\frac{x^2}{2\sigma_h^2}+\frac{y^2}{2\sigma_v^2})} e^{-(\frac{s_x^2}{2\sigma_{a_h}^2}+\frac{s_y^2}{2\sigma_{a_v}^2})}$$

where $\hat{s} = (s_x, s_y, 1) = (x/z, y/z, 1)$, the σ's are the beam sizes and angular spreads and $H(\nu)$ is frequency spectrum.

Brightness is Fourier transform of coherence:

$$B(\vec{r},\hat{s},\nu) = k^2 \frac{1}{(2\pi)^2}\int \Gamma(\vec{r},\vec{r}_1) e^{ik\hat{s}_\perp\cdot\vec{r}_1} d\vec{r}_1$$

Schell Model

Again the mutual coherence is

$$\Gamma(x_1,x_2) = \frac{I_0}{\sqrt{2\pi}\sigma_s} e^{-\frac{(x_1+x_2)^2}{8\sigma_s^2}} e^{-\frac{(x_2-x_1)^2}{2\delta_a^2}}$$

Giving a relation between angular spread and coherence length;

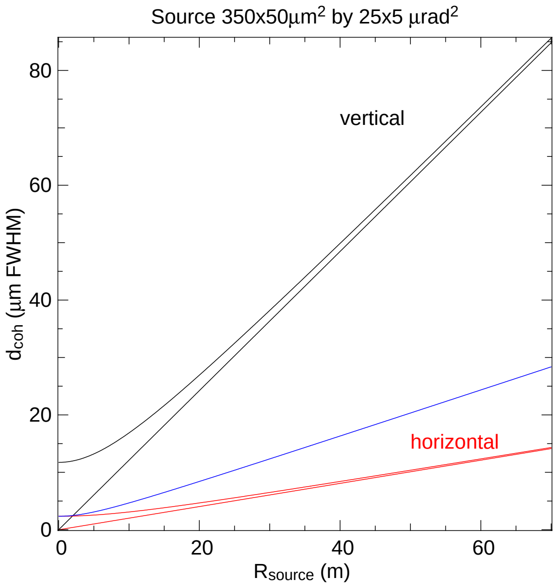
$$\rho_c^2 = \frac{1}{k^2\sigma_a^2 - \frac{1}{4\sigma_s^2}} = \frac{4\sigma_s^2}{4k^2\sigma_a^2\sigma_s^2 - 1}$$

Note, the condition $4k^2\sigma_a^2\sigma_s^2 \geq 1$ or $\sigma_a\sigma_s \geq \lambda/4\pi$ is simply the uncertainity principle.

APS Undulator A Coherence Lengths

Can propagate the partial coherence to experimental station and the coherence lengths are:

$$\Delta_i = \rho_i \sqrt{1 + \left(\frac{z}{k\sigma_i\rho_i}\right)^2}$$



Coherence lengths of an undulator source versus distance. The coherence lengths for an incoherent sources are plotted for comparison. Blue line is for smaller horizontal source.

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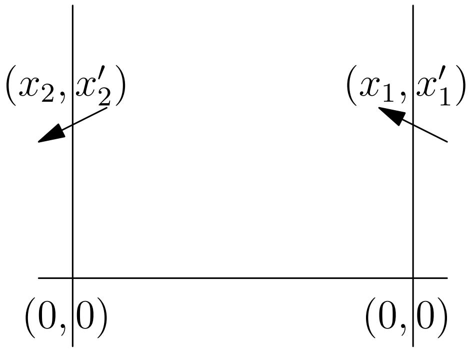
ABCD Matrices



From geometric optics, any ray is specified by a position x and angle x'. A ray at plane 2 comes from a a unique ray on plane 1.

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \mathbf{M} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

for some **M** an ABCD matrix.

If n_1 and n_2 are the indices of refraction of the medium at the input and output planes,

$$\det(\mathbf{M}) = AD - BC = \frac{n_1}{n_2} = 1 \quad \text{(for us)}.$$

ABCD Matrices

Each optical element has an ABCD matrix M_i. The ABCD matrix for the system is obtained by multiplication:

$$M = M_n \times M_{n-1} \times \cdots \times M_2 \times M_1. \quad (1)$$

ABCD Matrices

Element	Matrix	Remarks
Propagation in free space or in a medium of constant refractive index	$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$	d =length
Thin lens	$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$	Focal length f, $f > 0$, converging, $f < 0$, diverging
Reflection from a flat mirror	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Identity matrix
Refraction at a flat interface	$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$	n_1, initial refractive index, n_2, final refractive index
Reflection from a curved mirror	$\begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix}$	$R =$ radius of curvature, $R > 0$ for convex

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Gaussian Beams

Propagation of a paraxial Gaussian beam in the z direction gives:

$$E(x,y,z) = E_0 \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-ikz+i\psi(z)}}{2\sigma(z)} e^{-\frac{x^2+y^2}{4\sigma^2(z)}} e^{-ik\frac{x^2+y^2}{2R(z)}}.$$

with beam size is $\sigma(z)$, radius of curvature $R(z)$ and $\psi(z)$ is the Gouy phase factor. Defining $z = 0$ at the waist of this beam (width σ_0), gives:

$$\begin{aligned}
\sigma(z) &= \sigma_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \\
R(z) &= z + \frac{z_R^2}{z} \\
\psi(z) &= \tan^{-1}\left(\frac{z}{z_R}\right) \\
z_R &= \frac{4\pi\sigma_0^2}{\lambda}.
\end{aligned}$$

The last line defines the Rayleigh length (or Fresnel length).

Propagation of Coherence

Using Huygens-Fresnel, propagating $\Gamma(r_1, r_2)$ gives

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \int_S \int_S \Gamma(\mathbf{r}'_1, \mathbf{r}'_2, \tau) \frac{e^{ik(L_2 - L_1)}}{L_1 L_2} \Lambda_1^*(k) \Lambda_2(k) d^2\mathbf{r}'_1 \, d^2\mathbf{r}'_2,$$

where $L_i = \mathbf{r}_i - \mathbf{r}'_i$ and $\Lambda_i(k) = ik/2\pi$ for paraxial beams.

For an ABCD optical system (using 1D to simplify notation)

$$\Gamma(x_1; x_2, \tau) = \int\int \Gamma(x'_1; x'_2, \tau) \, K^*(x'_1; x_1) \, K(x'_2; x_2) \, dx'_1 dx'_2, \quad (2)$$

where

$$K(x'; x) = \sqrt{\frac{ik}{2\pi B}} e^{\frac{ik/2}{B}(Ax'^2 - 2x'x + Dx^2)} \quad (3)$$

is the propagation factor. Phase factors cancel in $K^* K$.

Propagation of Coherence

If $B = 0$ then propagation is between *conjugate* planes, object and image planes). For $B = 0$ and $A = D = 1$,

$$\Gamma(x_1'; x_2', \tau) = \Gamma(x_1'; x_2', \tau) P^*(x_1') P(x_2'), \tag{4}$$

where

$$P(x) = e^{\frac{ik}{2} C x^2}$$

is a pure phase factor. For instance for a thin lens of focal length f, $C = -1/f$.

Propagation of Coherence

For simple propagation ($A = D = 1,\ C = 0$ and $B = z$)

$$\Gamma(x_1, x_2; z) = \frac{I_0}{\sqrt{2\pi}\sigma_s \Delta(z)} e^{-\frac{(x_1+x_2)^2}{8\sigma_s^2 \Delta^2(z)}} e^{-\frac{(x_2-x_1)^2}{2\delta_a^2 \Delta^2(z)}}$$

where

$$\Delta(z) = \sqrt{1 + \left(\frac{z}{k\sigma_s\delta_a}\right)^2} = \sqrt{1 + \left(\frac{\sigma_a}{\sigma_s}\right)^2 z^2}$$

The length $z_R = k\sigma_s\delta_a = \sigma_s/\sigma_a$ is a generalization of the Rayleigh or Fresnel length to include partial coherence.

Propagation of Coherence

The intensity at z has a width

$$\sigma_s^2(z) = \sigma_s^2\Delta^2(z) = \sigma_s^2 + \frac{z^2}{k^2\delta_a^2} = \sigma_s^2 + \sigma_a^2 z^2$$

deriving the angular spread in another way.

Unravelling back we get $\rho_c(z) = \rho_c\Delta(z)$ or

$$\rho_c^2(z) = \rho_c^2 + \frac{z^2}{k^2\sigma_s^2}\left(1 + \frac{\rho_c^2}{2\sigma_s^2}\right).$$

At large z and small coherence length at the source,

$$\rho_c(z) = \lambda z / 2\pi\sigma_s$$

For Paraxial beams, with any ABCD optical system, the coherence length is a constant fraction of the beam's dimension.

(Note exception for slits.)

Propagation of Coherence

In general, for optical system ABCD:

$$\Delta_i = \sqrt{\left(A + \frac{B}{r_i}\right)^2 + \left(\frac{B}{z_R}\right)^2}, \tag{5}$$

and the radius of curvature is

$$R_i = \operatorname{sgn}(B) \frac{\left(A + \frac{B}{r_i}\right)^2 + \left(\frac{B}{z_R}\right)^2}{\left(A + \frac{B}{r_i}\right)\left(C + \frac{D}{r_i}\right) + \left(\frac{1}{z_R}\right)^2 BD}. \tag{6}$$

This completely specifies the coherence for any beam.

Why Coherence?

Coherence allows one to measure the dynamics of a material (X-ray Photon Correlation Spectroscopy, XPCS).

$$\langle I(\vec{Q},t)I(\vec{Q}+\delta\vec{\kappa},t+\tau)\rangle = \langle I(Q)\rangle^2 + \beta(\vec{\kappa})\,\frac{r_0^4}{R^4}V^2 I_0^2 \left|S(\vec{Q},t)\right|^2$$

where the coherence part is:

$$\beta(\vec{\kappa}) = \frac{1}{V^2 I_0^2}\int_V\int_V e^{i\vec{\kappa}\cdot(\vec{r}_2-\vec{r}_1)}\left|\Gamma(\vec{0},\vec{r}_2^{\perp}-\vec{r}_1^{\perp},\frac{\vec{Q}\cdot(\vec{r}_2-\vec{r}_1)}{\omega_0})\right|^2 d\vec{r}_1 d\vec{r}_2$$

and $\beta(\vec{0}) \approx \frac{V_{coherence}}{V_{scattering}}$ with widths $\lambda/V^{\frac{1}{3}}$

Reference: M. Sutton, Coherent X-ray Diffraction, in **Third-Generation Hard X-ray Synchrotron Radiation Sources: Source Properties, Optics, and Experimental Techniques**, edited by. Dennis M. Mills, John Wiley and Sons, Inc, New York, (2002).

Coherence in Diffraction

Using

$$|\Gamma(x_1, z_1, x_2, z_2, t)|^2 = V^2 \, \langle |E_i|^2 \rangle^2 \, e^{\frac{-(x_2-x_1)^2}{\xi_h^2}} e^{\frac{-(z_2-z_1)^2}{\xi_v^2}} e^{-2t/\tau}.$$

ξ_h, ξ_v transverse coherence lengths
$\xi_l = c\tau = \lambda^2/(\pi\Delta\lambda) = 2\lambda/(k_0\Delta\lambda)$: longitudinal coherence length

The coherence factor is thus

$$\beta \;=\; \beta(0) \;=\; \int_V \int_V e^{\frac{-(x_2-x_1)^2}{\xi_h^2}} e^{\frac{-(y_2-y_1)^2}{\xi_v^2}} e^{-2t/\tau}.$$

where V is the diffraction volume.

Coherence in Diffraction

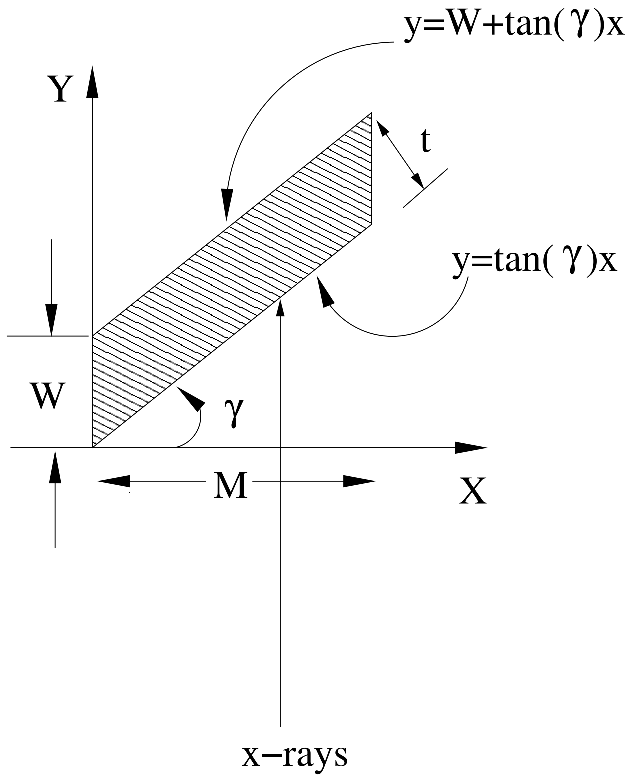
The time difference is

$$t = \vec{Q} \cdot (\vec{r}_2 - \vec{r}_1) / ck_0$$

and thus

$$\begin{aligned} t/\tau &= \frac{\vec{Q} \cdot (\vec{r}_2 - \vec{r}_1)}{ck_0} \times \frac{k_0 c \Delta\lambda}{2\lambda} \\ &= \frac{\Delta\lambda}{\lambda} Q \sqrt{1 - \frac{Q^2}{4k_0^2}} \Delta x - \frac{\Delta\lambda}{2\lambda} \frac{Q^2}{k_0} \Delta y \\ &= (A\Delta x + B\Delta y)/2 \end{aligned}$$

Coherence in Diffraction



Coherence in Diffraction

So define $\beta = \beta_z \beta_r$
where

$$\beta_z = \frac{1}{M^2}\int_0^M dz_1 \int_0^M dz_2 e^{\frac{-(z_2-z_1)^2}{\xi_v^2}}$$
$$= \frac{\xi_v^2}{M^2}\left[\frac{M}{\xi_v}\sqrt{\pi}\mathrm{erf}(\frac{M}{\xi_v}) + e^{\frac{-M^2}{\xi_v^2}} - 1\right]$$

and

$$\beta_r = \frac{1}{(WL)^2}\int_0^L dx_1 \int_0^L dx_2 \int_0^W dy_1 \int_0^W dy_2 e^{\frac{-(x_2-x_1)^2}{\xi_h^2}} e^{-|A(x_2-x_1)+B(y_2-y_1)|}$$
$$= \frac{2}{(WL)^2}\int_0^L dx(L-x)\int_0^W dy(W-y)e^{\frac{-x^2}{\xi_h^2}}\left[e^{-|Ax+By|} + e^{-|Ax-By|}\right].$$

This can be straight forwardly evaluated numerically.

Coherence in Diffraction

$$\delta_{zz}^{-2} = \frac{1}{\beta V^2 I_0^2} \int_V \int_V (z_2 - z_1)^2 \left| \Gamma(\vec{0}, \vec{r}_2^{\perp} - \vec{r}_1^{\perp}, \frac{\vec{Q} \cdot (\vec{r}_2 - \vec{r}_1)}{\omega_0}) \right|^2 d\vec{r}_1 d\vec{r}_2$$

and

$$\delta_{xx}^{-2} = \frac{1}{\beta V^2 I_0^2} \int_V \int_V (x_2 - x_1)^2 \left| \Gamma(\vec{0}, \vec{r}_2^{\perp} - \vec{r}_1^{\perp}, \frac{\vec{Q} \cdot (\vec{r}_2 - \vec{r}_1)}{\omega_0}) \right|^2 d\vec{r}_1 d\vec{r}_2$$

and

$$\delta_{rr}^{-2} = \delta_{xx}^{-2} \cos^2 2\theta + \delta_{yy}^{-2} \sin^2 2\theta$$

where δ_{rr} is the speckle width in the scattering plane (radial).

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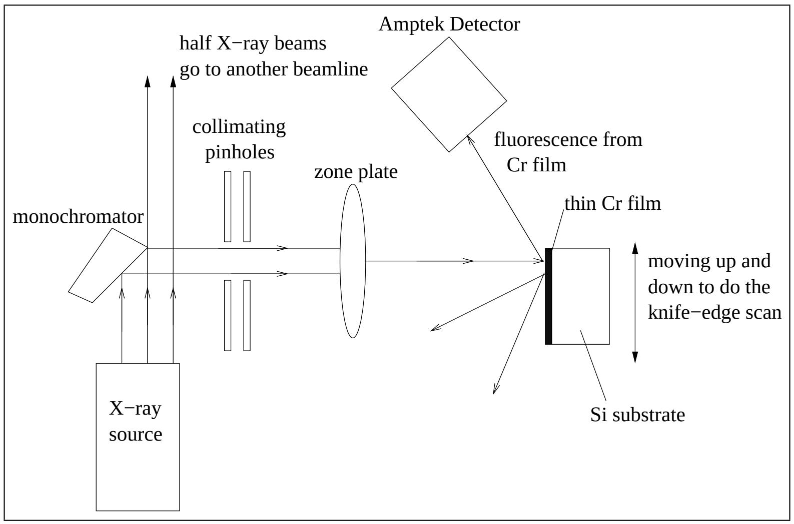
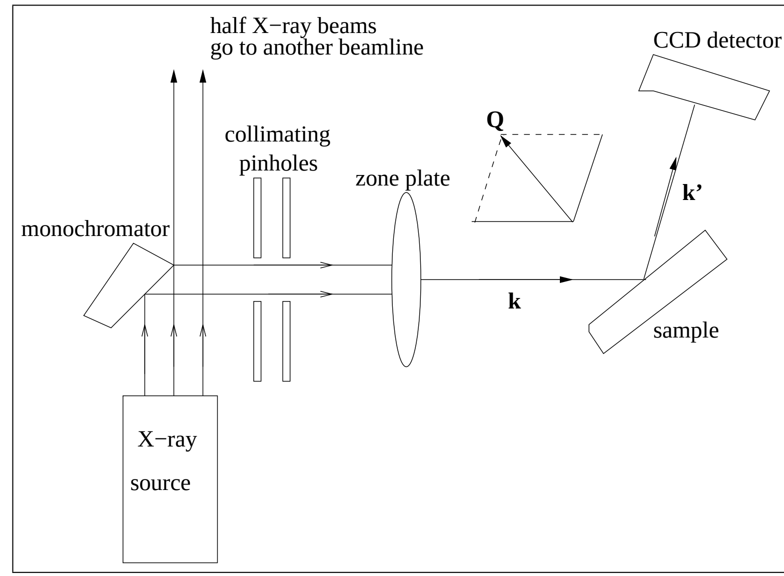
X-ray Setup, Beamline 8-ID-E (APS)

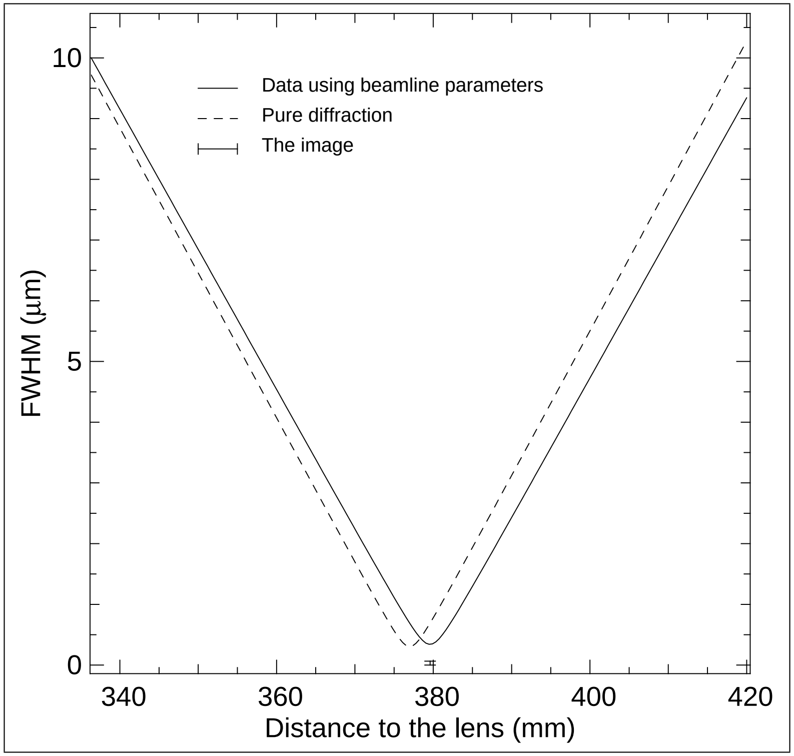


Knife edge

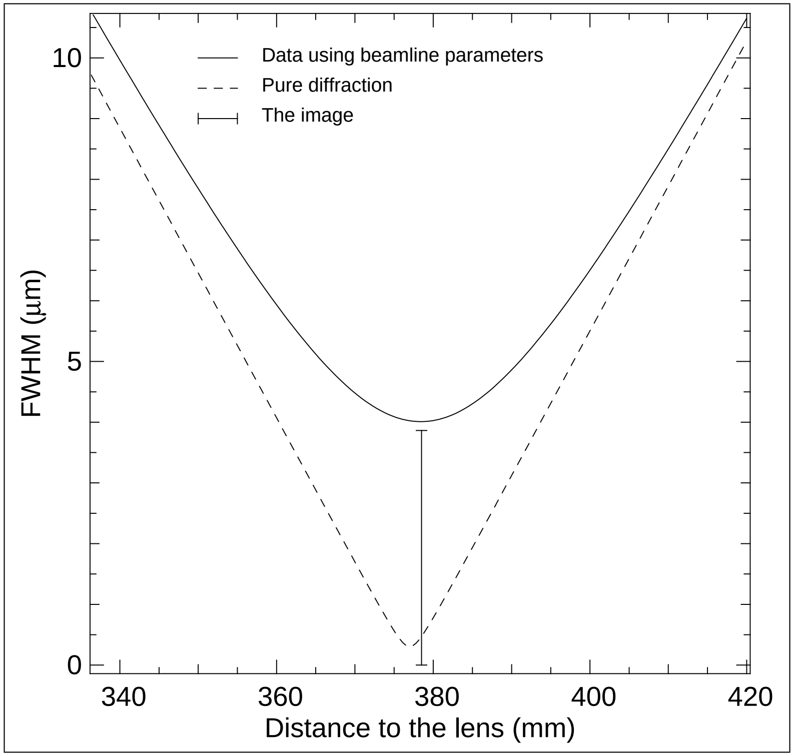


Diffraction

Fresnel lens propagation

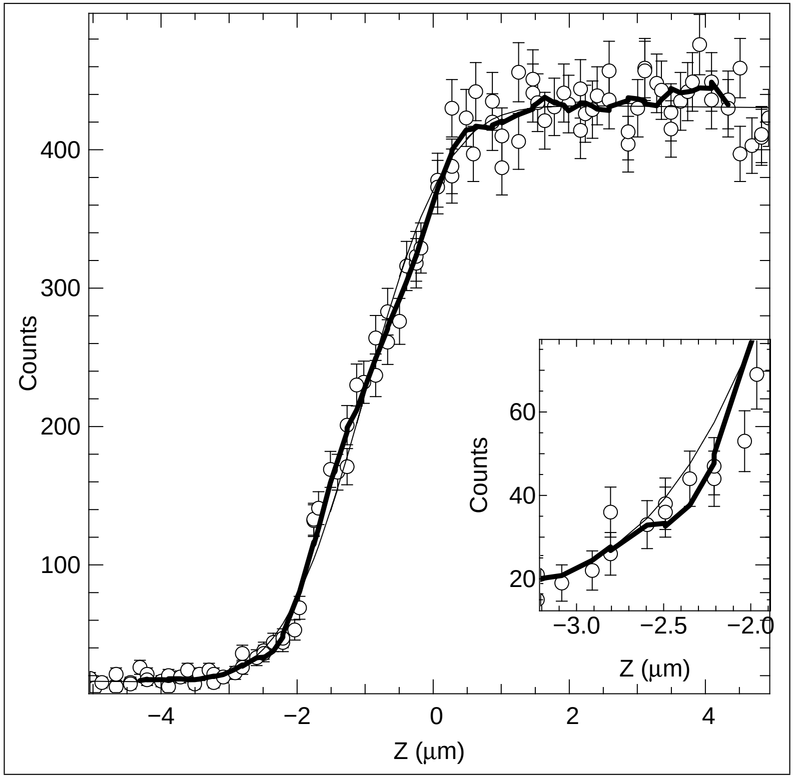


Vertical beam size.

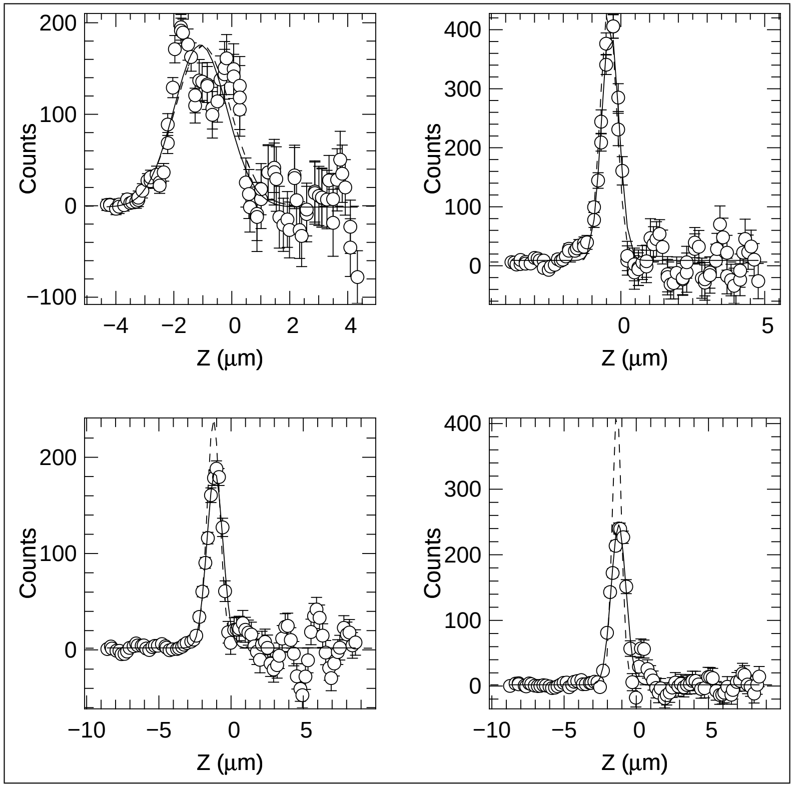


Horizontal beam size.

Fitting Knife-edge Scans

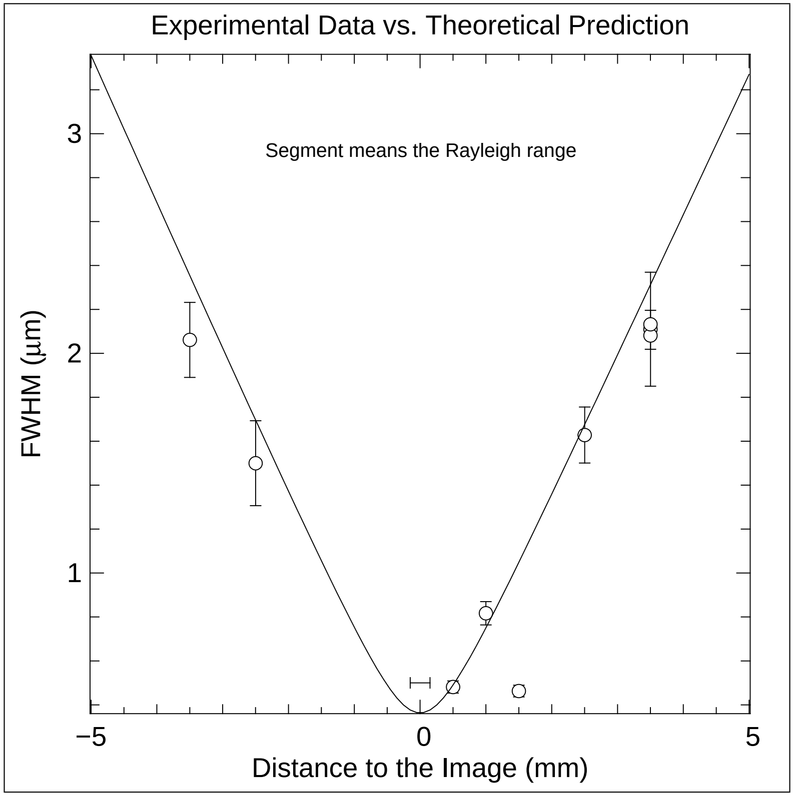


Smoothing for derivative.

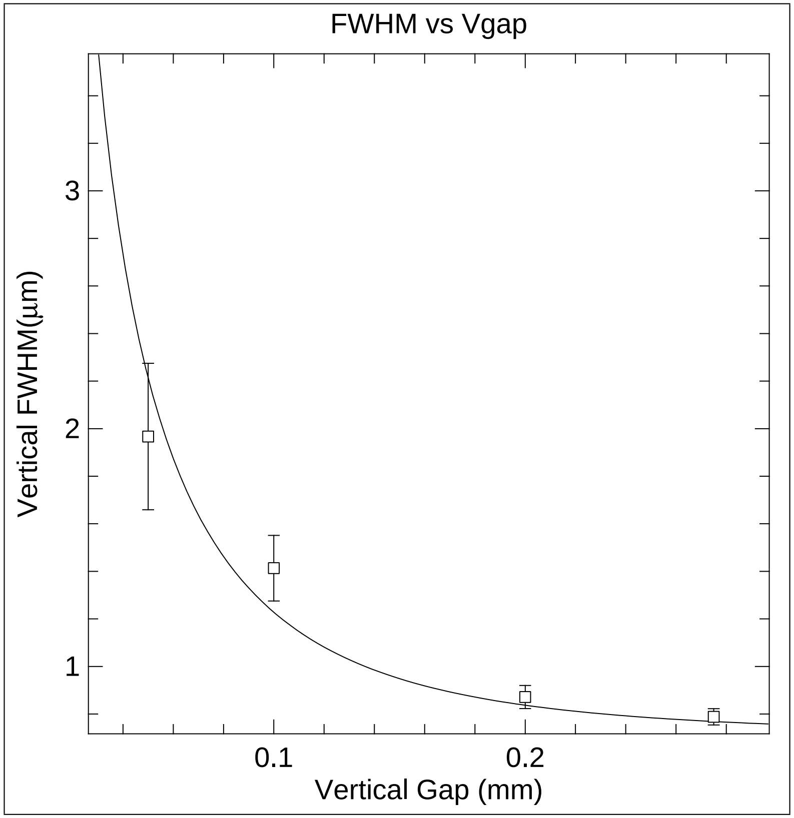


Directly Fit Derivative of Scan

Fitting Knife-edge Scans

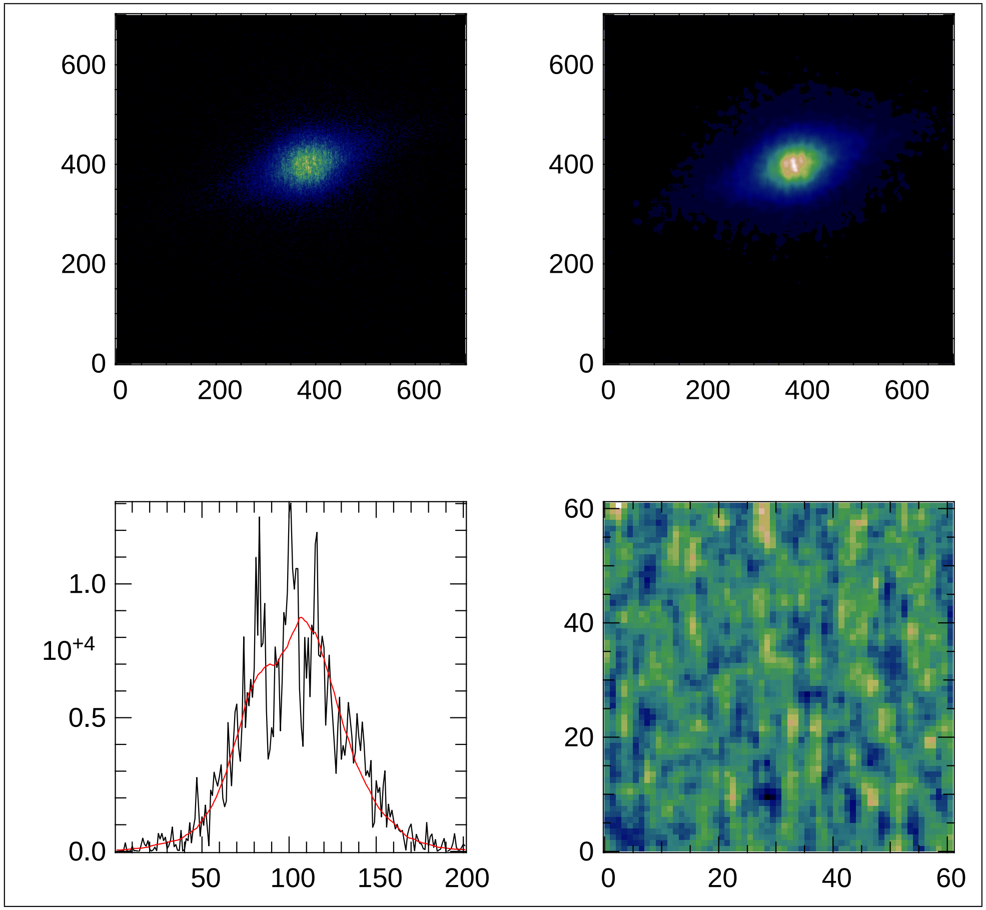


Vertical beam size.



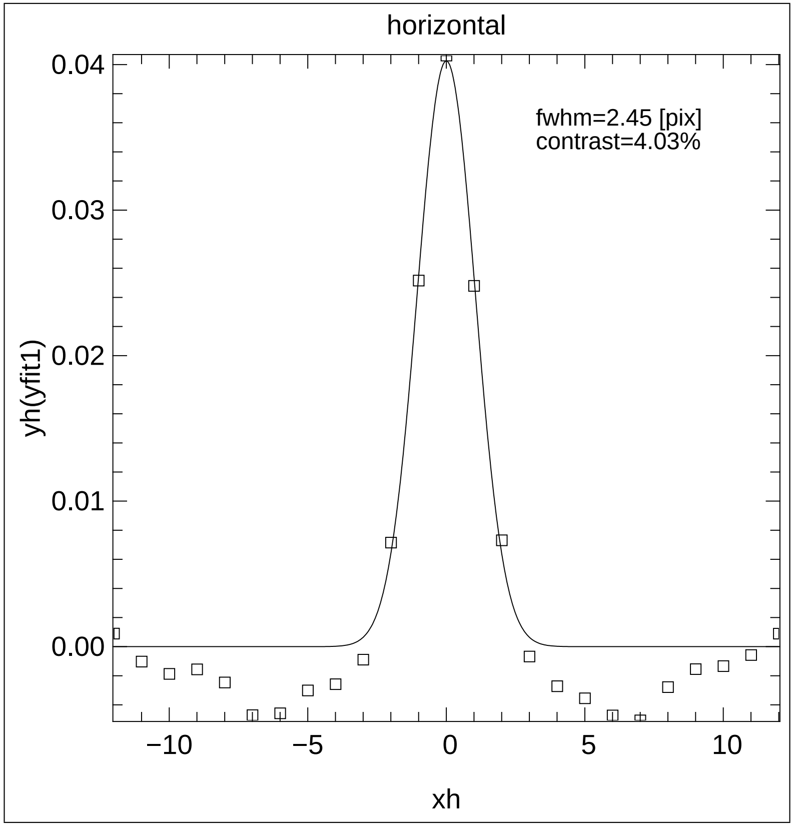
Size versus gap.

Coherence Measurement

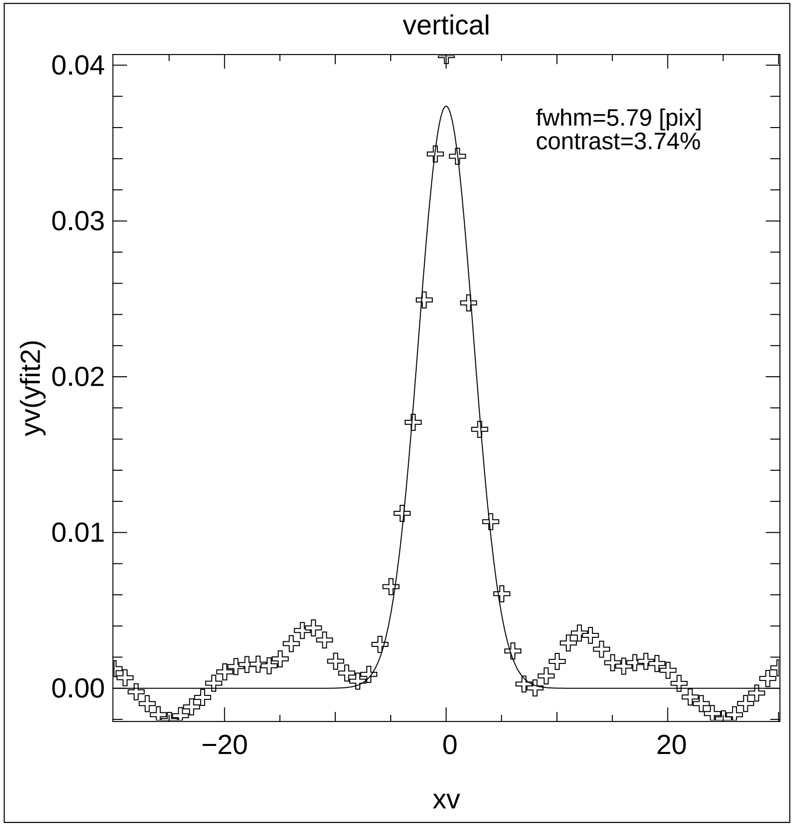


Cross-correlation Speckle Analysis. Fe_3Al $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $Q = 1.5 Å^{-1}$

Coherence Measurement

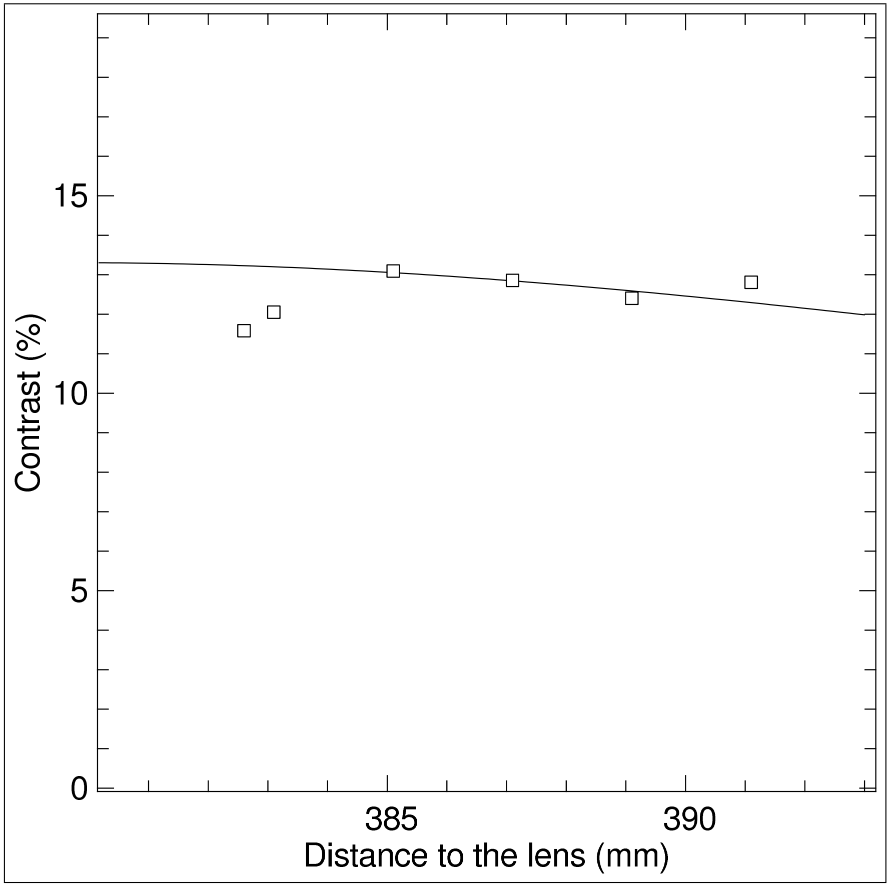


Horizontal speckle size and contrast.



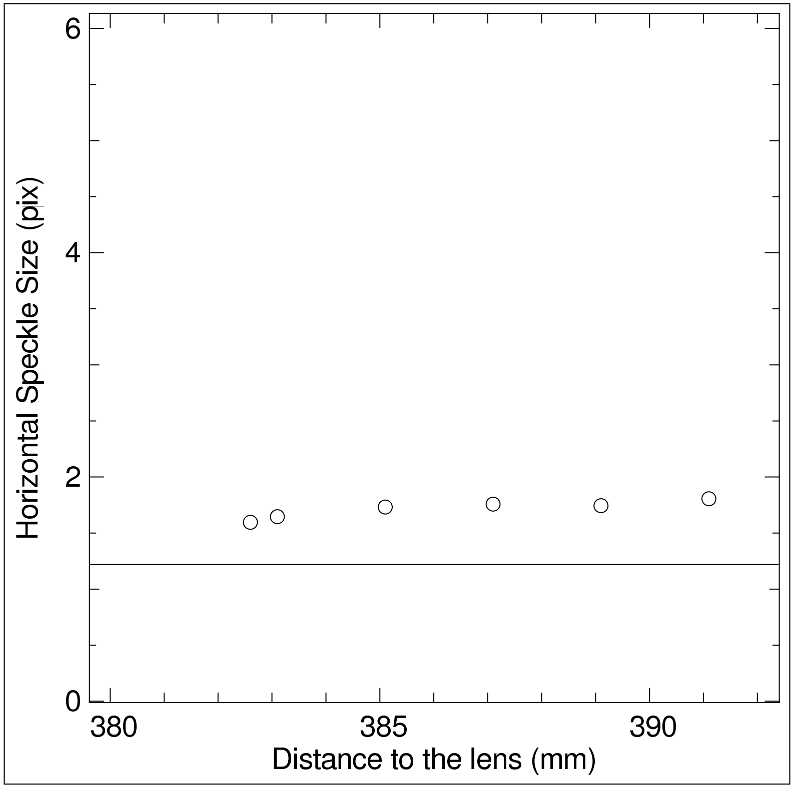
Vertical speckle size and contrast.

Speckle Contrast Fe_3Al $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

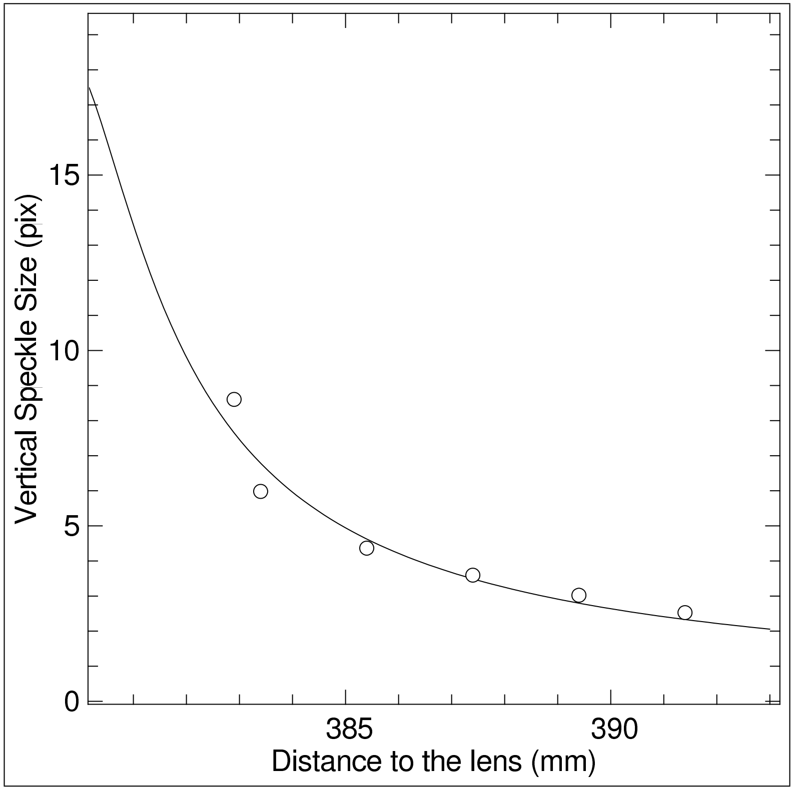


Cross-correlation Speckle Analysis.

Speckle Contrast Fe_3Al $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$



Horizontal speckle size and contrast.



Vertical speckle size and contrast.

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Signal to Noise

Signal is $g_2 - 1 = \beta$ and variance of is $var(g_2) \sim 1/(\bar{n}^2 N)$. So:

$$\begin{aligned}\frac{s}{n} &= \beta \bar{n} \sqrt{N} \\ &= \beta I \tau \sqrt{\frac{t_{total}}{\tau} N_{speckles}} \\ &= \beta I \sqrt{\tau t_{total} N_{pixels}}\end{aligned}$$

Note 1: This is linear in number of photons (as opposed to $\sqrt{\bar{n}}$).
Note 2: For fixed $s/n \sim \alpha I \sqrt{\tau/\alpha^2}$. Thus an α-fold increase in intensity is an α^2-fold increase in time resolution. Need very fast detectors.

Reference: Area detector based photon correlation in the regime of short data batches: data reduction for dynamic x-ray scattering, D. Lumma, L.B. Lurio, S.G.J. Mochrie, and M. Sutton, Rev. Sci. Instr. **71**, 3274-3289 (2000).

Signal to Noise

More explicitly:

$$\begin{aligned}
\frac{s}{n} &\approx \beta\, B_0 dx dx' dy dy' \frac{\Delta E}{E}\ \frac{1}{V}\frac{d\sigma}{d\Omega} L\ \sqrt{N_{sp}} \\
&\approx \beta B_0 f_x f_y\, \lambda^2\, \frac{\Delta E}{E}\ \frac{1}{V}\frac{d\sigma}{d\Omega} f_z \frac{\lambda^2}{\Delta\lambda}\ \sqrt{N_{sp}} \\
&\approx \frac{1}{max(1,f_i)^3} B_0 f_x f_y f_z\, \lambda^2\, \frac{\Delta\lambda}{\lambda}\ \frac{1}{V}\frac{d\sigma}{d\Omega}\frac{\lambda^2}{\Delta\lambda}\ \sqrt{N_{sp}} \\
&\approx B_0\, \lambda^3 \frac{1}{V}\frac{d\sigma}{d\Omega}\sqrt{N_{sp}} \\
&\approx f B_0\, \lambda^3 \frac{1}{V}\frac{d\sigma}{d\Omega}\sqrt{N_{sp}}\quad (if any f_i < 1).
\end{aligned}$$

Note: should be a $\lambda^3/8$ as normally use $\lambda/2$.

Conclusions

1. Except for slits coherence length scales as beam size.
2. Partially coherent Gaussian beams are parameterized by two parameters, $\Delta(z)$ how size scales and R_c the radius of curvature of the beam (diverging or converging).
3. Diffraction mixes transverse and longitudinal coherence.
4. ABCD optics give a good scaling relation on how coherence varies along the beam.